NOTE ON A THEOREM OF DILWORTH

by

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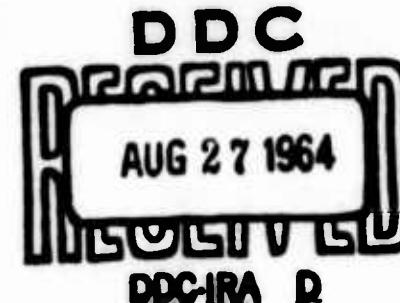
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SUMMARY

A theorem due to Dilworth on chain decompositions of partially ordered sets is shown to be a consequence of Menger's theorem in the theory of linear graphs.

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Let P be a finite partially ordered set with elements
$1, 2, \ldots, n$ and order relation denoted by ">." A <u>chain</u> in P
is a set of one or more elements $i_1, i_2, \ldots, i_k$ with
$i_1 > i_2 > \ldots > i_k$. A <u>decomposition</u> of P is a partition of
P into chains; a decomposition with the smallest number of chains
is <u>minimal</u>. Two members i,j of P are <u>unrelated</u> if neither
$i > j$ nor $j > i$. Dilworth [3] has proved that the number of
chains in a minimal decomposition of P is equal to the maximal
number of mutually unrelated elements of P.

Recently Dantzig and Hoffman [1] have formulated the problem
of finding a minimal decomposition of P as a transportation—
type linear programming problem, and have shown that Dilworth's
theorem follows from the duality theorem of linear inequality
theory. Our aim here is to show that Dilworth's theorem can
be deduced from Menger's theorem [5, p.244] concerning linear
graphs. That the two theorems are related is not surprising,
since Menger's theorem may also be viewed as a consequence of
the duality theorem applied to a programming problem of
transportation type [2].

We need Menger's theorem in only the following rather
special form. Let L be a linear graph with node set N and
suppose N is partitioned into fixed subsets N_1, N_2. An N_1, N_2 cut C
of L is a subset of N having the property that every arc joining
a node of N_1 to a node of N_2 has some node of C as endpoint,
and no proper subset of C has this property. An N_1, N_2 join
J of L is a set of arcs of L, each of which joins a node of
N_1 to a node of N_2, and no two of which have a node in common.
Menger's theorem, applied to L and the given N_1, N_2 partition,
states that $\max_J |J| = \min_C |C|$ (where $|S|$ denotes the number of ele-
ments in set S), the maximum being taken over all N_1, N_2 joins J,
the minimum over all N_1, N_2 cuts C. A proof of this particular
version of Menger's theorem has also been given by Egerváry [4].

We proceed to a deduction of Dilworth's theorem. Given
the partially ordered set $P = \{1, 2, \ldots, n\}$, let L be the linear
graph consisting of 2n nodes, labeled $a_1, \ldots, a_n$, $b_1, \ldots, b_n$,
and arcs defined from P by the rule: If $i > j$, then $a_i b_j$ is
an arc of L; these are all the arcs of L. Let $N_1 = \{a_1, \ldots, a_n\}$,
$N_2 = \{b_1, \ldots, b_n\}$. Henceforth all joins and cuts are relative
to N_1, N_2.

Lemma 1. Corresponding to any join J of L, there is a
decomposition D of P with $|J| + |D| = n$.

Lemma 2. Corresponding to any cut C of L, there is a set
U of mutually unrelated elements of P with $|C| + |U| = n$.

Note that Dilworth's theorem follows from the lemmas and Menger's theorem. For let $\hat{J}$ be a maximal join in L, $\hat{C}$ a minimal cut, and let $\hat{D}$, $\hat{U}$ be their respective correspondents in P. By Menger's theorem, $|\hat{J}| = |\hat{C}|$; hence, by the lemmas, $|\hat{D}| = |\hat{U}|$. But $|U| \leq |D|$ for any U and D, since two unrelated elements can not belong to the same chain. Hence $\max_U |U| = \min_D |D|$.

Proof of Lemma 1. Let $J = \left\{ a_{i_1} b_{i_2}, a_{i_3} b_{i_4}, \ldots, a_{i_{2k-1}} b_{i_{2k}} \right\}$.

Thus

$$i_1 > i_2, \quad i_3 > i_4, \quad \ldots, \quad i_{2k-1} > i_{2k}$$

in P, and we may put these together to form chains in the obvious way. These chains will be disjunct, since J is a join in L. By adding to these, as one-element chains, all indices from $1, 2, \ldots, n$ which do not already occur, a decomposition D of P is obtained. If the length of the ith chain in D is ℓ_i, then

$$n = \sum_{i=1}^{|D|} \ell_i = \sum_{i=1}^{|D|} (\ell_i - 1) + |D| = |J| + |D|.$$

Proof of Lemma 2. Let $C = \left\{ a_{i_1}, \ldots, a_{i_k}, b_{j_1}, \ldots, b_{j_m} \right\}$. The elements of the set I of indices $i_1, \ldots, i_k, j_1, \ldots, j_m$ are all distinct, for suppose $i_1 = j_1$, say. Since C is a cut, there is an $a_r \notin C$ with $a_r b_{j_1}$ an arc of L; similarly there is a $b_s \notin C$ with $a_{i_1} b_s$ an arc of L. Then, by the transitivity of the ordering and the assumption that

$i_1 = j_1$, it follows that $a_r b_s$ is an arc of L. This contradicts the fact that C is a cut, and thus implies that the elements of I are all distinct. Now let U be the complement of the set I in $1, 2, \ldots, n$. Since C is a cut, the elements of U are mutually unrelated in P, and $n = |U| + |C|$.

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